CHILDREN'S MENTAL MULTIPLICATION AND DIVISION STRATEGIES

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This paper reports changes in children's mental computation solution strategies for multiplication and division applied word problems (involving 1, 2 and 3-digit numbers combinations). The study followed 95 Queensland children from Year 4 through to the end of Year 6. The children's responses showed a development from simple counting to use of derived or known facts for small number combinations, and from counting to quite complex and creative strategies to algorithmic procedures for large number combinations. There was some evidence of instructional effects in the increased use of the taught algorithms, the continued use of counting strategies. There was, at times, sustained use of wholistic.

Interest in mental computation as an important computational method for numbers of two or more digits is not new. However, its significance is now seen in terms of its contribution to number sense as a whole; for example, as a "*vehicle* for promoting thinking, conjecturing, and generalizing based on conceptual understanding rather than as a set of skills that serve as an *end* of instruction" (Reys & Barger, 1994, p. 31). To achieve this contribution, it appears necessary to develop proficiency in mental computation through the acquisition of self-developed or spontaneous strategies rather than memorisation of procedures (Kamii, Lewis, & Livingston, 1993; Reys & Barger, 1994). Mental computation in this form features in various models of computation (e.g., National Council of Teachers of Mathematics, 1989; Trafton, 1994), although usually in combination with written, and calculator methods.

There is research evidence that children can use self-developed strategies to efficiently and effectively solve mental multiplication and division problems of two or more digits, even before instruction (e.g., Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997). Even in studies of children's solution strategies for more difficult multiplication and division word problems (Murray, Olivier, & Human, 1994), some self-developed strategies have been used (e.g., repeated addition, decomposition and compensation for multiplication; and repeated subtraction, use of multiplication and partitioning for division). There is also evidence for the negative effect of traditional algorithm instruction on efficient mental strategies for multiplication and division examples. For example, Kamii et al. (1993) reported that 60% of third graders who had not been taught the traditional multiplication algorithm were able to mentally solve 13 x 11 (by thinking 13x10=130, 130+13=143); a problem which, in contrast, was only successfully mentally solved by 15% of fourth graders who had been taught the algorithm.

Research has also indicated that performance in mental multiplication and division problems is influenced by the semantic structure of the word problem, with some problems being more difficult than others (e.g., cartesian product multiplication was found to be poorly attempted compared with other types of multiplication - Mulligan, 1992). However, the solution strategy used did not always reflect the semantic structure, particularly as children progressed (Mulligan, 1992; Mulligan & Mitchelmore, 1997; Murray et al., 1994); for instance, children rarely used sharing for partition division (Mulligan, 1992). Similar results have been found for subtraction (Heirdsfield & Cooper, 1996).

Many researchers have categorised children's solution strategies for multiplication and division word problems (e.g., Anghileri, 1989; Boero, Ferrari, & Ferroro, 1989; Bryant, Morgado, & Nunes, 1993; Carpenter et al., 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan, 1992; Mulligan & Mitchelmore, 1997). Most of this research has been limited to small number combinations and, therefore, has categorised strategies as counting types (Mulligan & Mitchelmore, 1996). Some research has focused on more complex number combinations, describing strategies in detail (Murray et al., 1994). However, little research has looked across all number combinations.

This paper reports on Years 4 to 6 children's responses to six multiplication and division word-problem tasks which formed part of an Australian Research Council funded five-year longitudinal study of Years 2 to 6 Queensland children's mental strategies for the four operations. Students were tracked over the three years from simple 1 by 1-digit to more difficult 2 by 2-digit multiplication and 1 by 1-digit to 3 by 2-digit division word problems. Further, the study traced strategy changes from pre-instruction in multiplication and division terminology and notation (for some children), through a period of number-fact instruction, and finally until children were taught the written standard algorithms for 2-digit by 2-digit multiplication and 2 and 3-digit by 1-digit division. It also differed from previous research in that the emphasis was not on the semantic structure of the problems; rather, the emphasis was on identifying strategy choice for simple semantic structure and increasingly difficult number combinations.

Method

Subjects. The subjects were 95 children from 14 schools (Independent, Catholic and State). The schools were representative of differing socioeconomic backgrounds. The children had been chosen, when in Year 2, by their teachers to comprise one third of each of above average, average, and below average ability. During the study on which this paper reports, the children progressed through Years 4, 5 and 6.

The Queensland mathematics syllabus advocates that children be introduced to the concepts of multiplication and division in Year 2, the multiplication symbol (up to 9x9=81) in Year 3, the standard written multiplication algorithm (2 by 1-digit) and the division symbol (up to 81÷9=9) in Year 4, and the standard written multiplication algorithm (2 by 2-digits) and the standard partition written division algorithm (2 by 1-digit) in Year 5. Although schools generally followed the Queensland syllabus, there were classes that had not been formally introduced to the concepts of multiplication and division by Year 4.

Instrument. The instrument used was Piaget's clinical interview technique. The tasks, reported in this paper, comprised three equal grouping multiplication word problems (5x8, 5x19, 25x19) and two partition (24÷4, 100÷5) and one quotition division (168÷21) word problems. The six tasks represented a cross section of the possible multiplication and division problems and were the most frequently attempted problems in the larger study. They involved contexts common to children (money, lollies, and children in classes). They were given in picture form (the child listened as the interviewer said the problem); no algorithmic exercises were presented. The numbers were chosen and the pictures used in the hope that it would maximise the use of children's own invented strategies, and minimise the use of the traditional written algorithm.

Procedure. The students were interviewed in the second and fourth terms of Years 4, 5 and 6. They were withdrawn from the classroom and interviewed individually in a separate room. The interviews lasted for a maximum of 20 minutes and were videotaped. The word problems were presented visually in the form of pictures, and orally as the interviewer verbalised the task. Although all tasks were presented to all children, not all children were able to attempt every task. If the children attempted a task, further questions were asked to probe for the strategy they used.

Results

Strategy categories. The videotapes were viewed, children's responses were analysed for commonalities in relation to the procedures identified in the literature; and a list of initial strategies developed. Then, the responses of each child for each task in each interview were classified in terms of these strategies and recorded for each interview. Finally, the calculation strategies were considered carefully, and after discussion among the researchers, five strategy categories were identified for each of multiplication and division (see Table 1). All responses were then coded using these strategy categories, and the results were analysed for trends across the three years.

Table 1. *Mental multiplication and division strategy categories*

Category	**Description**	**Examples**
Multiplication *Counting* (CO)	Any form of counting strategy, skip counting forwards and backwards, repeated addition and subtraction, and halving and doubling strategies.	5x8: 5, 10, 15, … 5x8: double 5, double 16, +8.
Basic fact (BF)	Using a known multiplication or division fact or a derived fact.	5x8: 10x8=80, so 5x8=40.
RL separated (RLS)	Numbers are separated into place values, then proceed right to left.	5x19: 5x9=45=40+5, 5x10=50, 50+40=90, 95.
LR separated (LRS)	Numbers are separated into place values, then proceed left to right.	5x19: 5x10=50, 5x9=45, 50+45=95.
Wholistic (WH)	Numbers are treated as wholes.	5x19: 5x20-5=100-5=95. 25x19: 4x25=100, 4x4=16, 4x100=400, add 3x25(75), so 475.
Division *Counting* (CO)	Any form of counting strategy, skip counting forwards and backwards, repeated addition and subtraction, and halving and doubling strategies.	24÷4: 4, 8, 12,… 24÷4: half of 24, half of 12.
Basic fact (BF)	Using a known division fact or a derived fact.	24÷4: 4x?=24. 6 24÷4: 5x4=20, so 6x4=24.
LR separated (LRS)	Numbers are separated into place values, then proceed left to right.	100÷5: 10÷5=2, 0÷5=0, 20.
RL separated (RLS)	Numbers are separated into place values, then proceed right to left.	100÷5:, 0÷5=0, 10÷5=2, 20.

Wholistic (WH)	Numbers are treated as wholes.	100÷5: 100÷10=10, 10x2=20. 168÷21: 5x21=100 ∴ 5x21=105, about 60 left, 3x20=60 ∴ 3x21=63, 63+105=168, ans. 5+3=8.

General trends within each task

The results for multiplication are presented in Table 2 and, for division, in Table 3. As would be expected, the percent of children attempting and correctly attempting the tasks increased across the six interviews. Further, the multiplication tasks were easier for the children as evidenced by the higher percentage attempted, and attempted correctly, than the division tasks.

Table 2. *Multiplication responses for Interviews 1 to 6 (n=95)*

Question	Interview	% attempting (% correct)	% attempting (% correct)				
			CO	BF	RLS	LRS	WH
5x8	1 (Year 4)	92 (71)	54 (34)	38 (37)			
	2 (Year 4)	97 (87)	23 (17)	74 (71)			
	3 (Year 5)	99 (87)	22 (13)	77(75)			
	4 (Year 5)	99 (93)	17 (15)	82 (78)			
	5 (Year 6)	100 (96)	6 (5)	94 (91)			
	6 (Year 6)	100 (97)	3 (1)	97 (96)			
5x19	1	26 (18)	1 (0)		11 (7)	4 (2)	10 (8)
	2	68 (51)	6 (2)		37 (30)	12 (7)	14 (12)
	3	72 (51)	4 (1)		33 (26)	17 (7)	18 (16)
	4	86 (73)	16 (7)		44 (42)	6 (5)	20 (18)
	5	96 (85)	7 (4)		58 (51)	7 (7)	23 (23)
	6	98 (85)	4 (0)		56 (51)	11 (8)	27 (26)
19x25	1	2 (2)	1 (1)				1 (1)
	2	16 (5)	2 (0)		7 (0)		6 (5)
	3	28 (12)	7 (1)		8 (1)	2 (1)	11 (8)
	4	78 (32)	9 (3)		37 (8)	5 (1)	26 (19)
	5	79 (35)	9 (2)		28 (7)	4 (1)	37 (24)
	6	85 (45)	8 (2)		34 (13)	4 (1)	38 (29)

For task 5x8, *Counting* was the initial dominant strategy (included skip counting in fives and near doubles, e.g., double 8, double 16, add 8). However, by Interview 2, the *Basic fact* strategy was dominant and reasonably accurate.

A low of 26% attempted task 5x19 in Year 4, while 98% attempted it by the end of Year 6. From the end of Year 4 to the end of Year 6, the *RL separation* strategy was dominant, with the *Wholistic* strategy being used half as much (surprisingly due to the ease by which it applies to 5x19 (5x20-5). The *LR separation* strategy was used by a significant minority and some children persisted in using the *Counting* strategy into the last interview.

Task 19x25 was attempted by only two children in Interview 1. One child counted in 25s, the other used a *wholistic* strategy ("10x25=250, another 250, take 25"). Both solutions resulted in correct answers. From there, the number of children attempting a solution increased across the interviews, until 85% attempted the problem in the last interview. However, only about half the solutions were correct. Most errors resulted from the application of the *RL separation* strategy (which is not surprising considering the memory load needed to remember all the interim calculations). Strategies that were more successful in giving correct answers included *Counting* (counting in 25's and grouping in 100's), *Wholistic* (20x25=500, 500-25=475), and even *LR separation* (10x25=250, 9x25=225, using 8x25=200 as known, 250+225=475).

For task 24÷4, the dominant strategy for all interviews was *Basic fact*. Most children reported knowing, "twenty-four divided by four is six, because four sixes are twenty-four." The other strategy used was *Counting* (halving, doubling, repeated addition, skip counting and sharing). A very small minority of children solved the problem by sharing one at a time (reflecting the semantic structure of the problem), while another minority used halving accurately.

Table 3 *Division responses for Interviews 1 to 6 (n=95)*

Question	**Interview**	**% attempting (% correct)**	**% attempting (% correct)**				
			CO	**BF**	**LRS**	**RLS**	**WH**
24÷4	1	68 (57)	12 (5)	57 (52)			
	2	84 (77)	10 (7)	75 (70)			
	3	88 (82)	8 (6)	80 (76)			
	4	97 (91)	13 (11)	84 (80)			
	5	96 (95)	3 (2)	93 (93)			
	6	97 (96)	4 (4)	93 (92)			
100÷5	1	56 (48)	11 (18)	18 (18)	2 (2)		25 (23)
	2	72 (60)	8 (2)	26 (25)	6 (5)		31 (27)
	3	79 (73)	6 (5)	34 (32)	2 (2)		37 (34)
	4	90 (76)	12 (6)	40 (38)	2 (2)		36 (30)
	5	85 (81)	4 (4)	44 (42)	2 (2)		35 (33)
	6	94 (90)	1 (1)	56 (54)	5 (5)		33 (31)
168÷21	1	1 (1)	1 (1)				
	2	6 (3)	2 (0)		2 (1)	1 (1)	1 (1)
	3	11 (8)	5 (5)			1 (0)	4 (3)
	4	38 (26)	23 (13)		5 (5)	1 (1)	7 (6)
	5	47 (36)	21 (13)		4 (2)	5 (5)	17 (16)
	6	61 (51)	27 (21)	1 (1)	2 (1)	2 (2)	28 (25)

For task 100÷5, accuracy levels were generally high, with *Wholistic* (ignoring the final zero, or using 4x25=100 or 10x10=100) and *Basic fact* strategies (knowing that 5x20=100 or 100÷5=20) popular throughout.

Like 19x25, task 168÷21 was attempted by a lower number of children, with less accuracy, throughout the interviews (one child in Interview 1 through to 61% by Interview 6) and elicited a greater variety of strategies than the other four tasks. The *Counting* strategies included skip counting, repeated addition and doubling, and persisted across all interviews (only one child used repeated subtraction). The *Wholistic* strategies included trial and error for multiplication (e.g., "21 times something is 168. 7? No. I'll try 8. Yes.", "1 times something is 8, 8 times 2 is 16, so it's 8."), trial and error for division (e.g., "Something goes into 16, 2 times, and into 8 once. That's 8."), and partitioning (e.g., "about 100 and the rest, because I know 5 x 20 = 100"). By Interview 4 (end of Year 5), some children attempted to solve the problem using *LR separation*. Interestingly, a handful of these children said that they wouldn't be able to attempt 168÷21, because they had not been taught how to divide with 2-digit divisors; yet prior to this interview, no such excuse was made for the inability to solve the problem.

Discussion

Strategy use and preferences. Strategy use across the six interviews was influenced by number combinations and students' available strategies. Tasks that were basic facts (5x8 and 24÷4) tended to be solved initially by the *Count* strategy and, then, later by the *Basic fact* strategy. Tasks that involved more complex numbers were initially solved by a greater variety of strategies. Across the interviews, the strategy category preferences of the children moved increasingly to the more efficient strategies, specifically the *Separation* and *Wholistic* categories, except when the task was related to a basic fact (e.g., 100÷5). For multiplication with 2-digit numbers, the tasks were solved increasingly by *RL separation* after Interview 2. For the division task with a 2-digit divisor (168÷21), *LR separation* began to be used, without success, in Interview 4.

There was little or no use of repeated subtraction or sharing one to one (contrary to recommendations for teaching division in Queensland). A sharing strategy was used by weaker students, generally unsuccessfully, seemingly because of the heavy load on working memory. The trial and error strategy (e.g., 4x?=24. 5?, check by skip count or doubles or basic fact; no, try 6.) was found to be more reliable and efficient (similar to Mulligan, 1992).

Instructional effects. During Years 4 and 5, the traditional written multiplication and division algorithms are introduced to children. Their procedures are similar to the *RL separation multiplication* and *LR separation division* strategy and, hence, should reinforce and reduce working memory load for these strategies. *Wholistic* strategies appear to be less complex mentally than separation strategies (requiring less working memory) because they do not require numbers in the different place-value positions to be remembered and operated on separately, as is required by separation strategies. The four tasks where numbers were 2-digits or more had numbers chosen so that the *Wholistic* strategies were applicable. For example, 5x19 is close to 5x20, as is 100÷5, while 19x25 is close to 20x25 (and involves 25 which is one-quarter of 100), and 168÷21 is close to 160÷20. Therefore, it seemed reasonable to predict that *Wholistic* should have been the most efficient mental strategy for these four tasks, that separation strategies should have been little used, and that the use of the *RL separation* strategy involved some component of instructional effect.

There is some evidence that there may be an instructional effect, at least for multiplication (similar to the findings of Cooper, Heirdsfield, & Irons, 1996, for addition and subtraction). There was a trend to the *RL separation* category in tasks 5x19 and 19x25, yet the use of the *Wholistic* strategy was a little more accurate (particularly for 19x25). For division, there was not the same strength of support for an instructional effect in the strategy trends. However, there was some extra support for an instructional effect in division in the comments of the children. In Interview 4 and with task 168÷21, some children would not attempt the task because they "had not been taught to do long division with two digit divisors". Previous to this, the children had been willing to "have a go" at many tasks they had not covered in their mathematics classes. It seemed that the teaching of the division algorithm had "coloured" their approach to arithmetic.

Conclusions

The findings of this study show children's changing accuracy and strategy preference for mental multiplication and division across three years during which they were introduced to written algorithms for these operations. The children improved in percentage attempting the tasks and accuracy. However, there was not the expected change to more sophisticated strategies. Children stayed with *Counting* and, where they could, *Basic fact* strategies, and there was some evidence of movement to strategies based on the written algorithms. There was growth in the use of the *Wholistic* strategies where it was appropriate, but not to the extent that might be predicted from the deliberate favouring of these strategies in the choice of numbers in the tasks. There was little use of strategies based on non-standard algorithmic procedures, which was different from addition and subtraction mental computation (Cooper et al., 1996)

In the world of computers and calculators, estimation appears to be a more useful human ability than correct written calculation. Estimation seems better served by trial and error strategies (one of the *Wholistic* strategies), particularly when it is used mentally (as it so often has to be in real world situations). This study shows that many children, by the end of Year 6, were able to use quite advanced *Wholistic* strategies for larger number combination multiplication and division. However, another (although less efficient) strategy for these larger number combinations was *Counting*. Considering the numbers involved, *Counting* was reasonably efficient, certainly more efficient for 168÷21, than *LR separation*.

There appears to be a need, in multiplication and division mental computation as well as estimation, for assistance to be given to children to use strategies different from those associated with traditional computation (e.g., trial and error and *Wholistic*, and, maybe, some forms of non-standard *separation*). This would seem to imply a reduction of emphasis on written algorithms for multiplication and division (even their removal from the syllabus), a growth in instruction time spent on arithmetical properties and alternative computational strategies, and a change to more child-centred and flexible approaches to teaching operations.

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